A (RELATIVELY) NONTECHNICAL DESCRIPTION OF MY RESEARCH

CHRISTOPHER K. ATKINSON

1. Research description

The main theme of my research program is studying the connections between shape and size in three-dimensional hyperbolic geometry. Since Thurston formulated the Geometrization Theorem in the 1970s, it has become clear that understanding hyperbolic geometry is central to understanding the geometry of three-dimensional spaces in general. I will first describe the ideas behind hyperbolic geometry and then continue on to describe three–dimensional spaces and their generalizations.

1.1. **Curvature and hyperbolic space:** The Euclidean plane and Euclidean three–dimensional space is the typical setting for high school geometry. Figures in a flat plane or three–dimensional space are the objects of study.

Whereas the Euclidean plane or Euclidean three-dimensional space is flat, the hyperbolic plane and hyperbolic three-dimensional space are negatively curved. Curvature is a quantification of how a space is curved. It is intuitively understandable by way of example in two dimensions. The surface of a ball is positively curved: if an ant stands on a tennis ball, it may observe that the ball curves downwards (from its perspective) in any direction. The hyperbolic plane has negative curvature. A physical example of an object with negative curvature is a curly kale leaf. If a tiny mite is walking around on a kale leaf, it will observe that in some directions, the leaf curves downwards while in other directions it curves upwards. If one tries to force a piece of the hyperbolic plane or a kale leaf to lay flat on a table, it will become clear that there is too much material around each point. This curvature is a property that is intrinsic to a surface. There is no way to deform the surface in space to completely remove the features arising from its curvature at all points simultaneously. The idea of a negatively curved space in three-dimensions is similar, although a bit more difficult to visualize.

An accurate model of the two–dimensional hyperbolic plane is obtained by starting with a disk of radius 1 and changing the way that lengths are measured so that the boundary of the disk is actually infinitely far from any point inside the disk. In this model, hyperbolic straight lines manifest themselves as arcs of circles that meet the boundary of the disk perpendicularly. The right–hand image in Figure 1 is a picture of the hyperbolic plane with three hyperbolic lines. This model is set up so that the angle that you see in the picture is the actual hyperbolic angle. The three lines in the image together bound a shaded triangle. Three–dimensional hyperbolic space has a similar model coming from a ball of radius 1. In three dimensions, the hyperbolic lines are again arcs of circles that meet the bounding sphere perpendicularly. Portions of spheres that intersect the bounding sphere at right angles are copies of the two–dimensional hyperbolic plane sitting inside of three–dimensional hyperbolic space.

In Euclidean geometry, there is not usually a meaningful connection between shape and size. For example, the volume of a cube is not determined by the fact that the figure is a cube. One must take some sort of length measurement before determining the volume of a cube. This is a consequence of the fact that Euclidean space can be rescaled by magnifying or shrinking figures, keeping their overall shape unchanged. In hyperbolic space, the situation is completely different.

The negative curvature of hyperbolic space gives it rigidity. By rigidity, I mean that if one knows the shape of a figure in hyperbolic space, then its size is determined. This implies that a

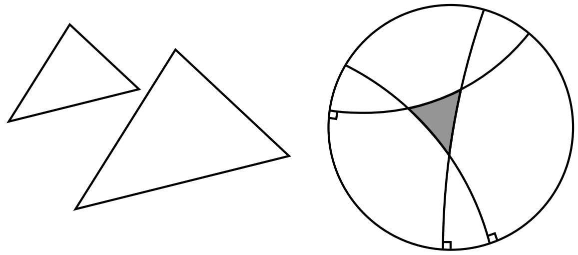

FIGURE 1. Euclidean triangles (left) may be rescaled, changing their size. Hyperbolic triangles (right) are rigid and cannot be rescaled.

figure typically cannot be rescaled in a way that preserves shape. For example, if a triangle in the hyperbolic plane has angles α, β, and γ, then its area is $\pi-\alpha-\beta-\gamma$ where π is the constant $3.141\ldots$ obtained by dividing the circumference of an ordinary circle by its diameter. This contrasts strongly with the situation in Euclidean geometry. As taught in high school geometry, knowing the three angles of a Euclidean triangle does not determine its side lengths so does not determine its area. See Figure 1.

A similar rigidity holds for hyperbolic spaces in higher dimensions. In fact, things tend to become more rigid as the dimension is increased. For example in all dimensions three and greater, not only is the volume of a polytope determined by its shape and angle measures, but there is only one such object up to rigid motions.

This rigidity of hyperbolic space persists in mathematical objects that are more complicated than polygons or polyhedra. A main goal of my research is the study of hyperbolic orbifolds and how their shape determines their size. An orbifold is a generalization of the idea of a manifold. I will now describe manifolds and then explain how to intuitively understand what an orbifold is.

1.2. **Manifolds and orbifolds:** A manifold is a mathematical object designed to model the notion of space. In our spatial universe, we tend to think that for a point at any given location, there is a region of space in which that point can potentially move. A manifold is a set of points so that every point is contained in a nice "neighborhood" of points, just as we imagine in our spatial universe.

Some examples of manifolds in two dimensions are the Euclidean plane, the surface of a ball, and the surface of a donut, also known as a torus. In each of these examples, each point is contained in a small disk of points in the manifold. Another way to say this is that an ant standing on a physical manifestation of one of these objects can walk a short distance in any direction and remain on the manifold. An example of a three-dimensional manifold is three-dimensional Euclidean space. Each point in three-dimensional space is contained in a three-dimensional neighborhood of points. A more complicated and illustrative example is known as the three-torus. To construct this example, take the set of points inside a cube and imagine that if a point moves through the front face of the cube, it is teleported to the back face. You can think of the front face as having been glued to the back face. Repeat this construction with the top and bottom faces and with the left and right faces. The result is a manifold. Living inside such a manifold would be a strange experience. Looking forward, you would see yourself from the back side. If you wave your left hand at this copy of yourself, you will see it also waving its left hand. Looking off at a skew angle, you'll see many copies of yourself arranged along a three-dimensional lattice. These aren't actually copies of yourself, though. You're seeing yourself from different angles as the light rays wrap around the cube. This manifold has Euclidean geometry since it can be "tiled" by an actual Euclidean cube. See the left–hand image of Figure 2 to see the earth sitting inside of a three–torus. Although this sounds far fetched, it is possible that we live in a universe having connectivity properties like the

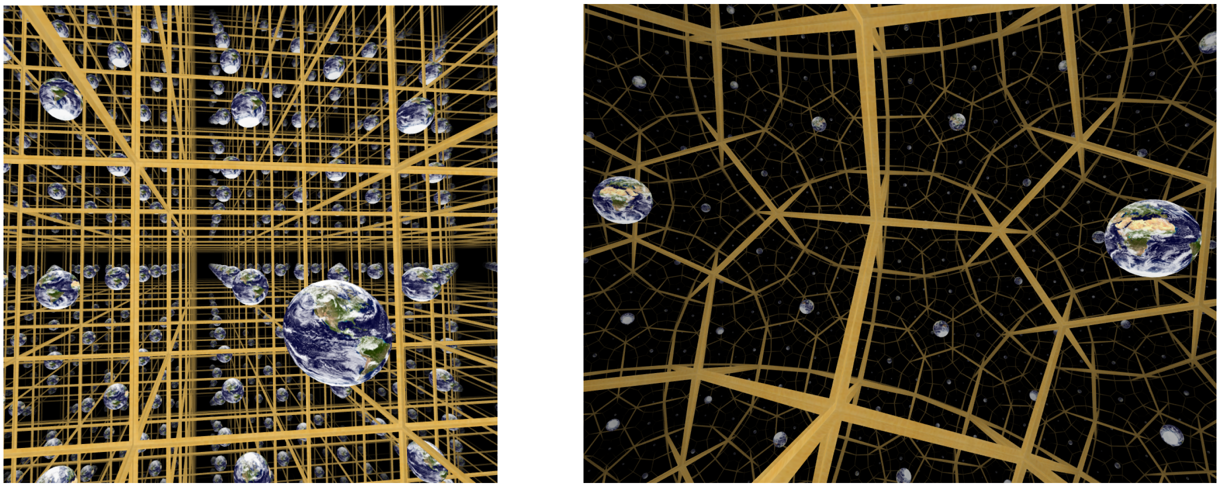

FIGURE 2. Earth in a 3–torus (left). Earth in a mirrored, right–angled, hyperbolic dodecahedron (right). Images captured from Jeff Weeks's program *Curved Spaces*. It's a fun program to play around with. Download here: `http://www.geometrygames.org/CurvedSpaces/index.html.en`

three–torus. It is theoretically possible that if we were to leave Earth on a spaceship and fly far enough in one direction, that we will eventually wrap around the universe and return to Earth.

An orbifold is a generalization of a manifold. This means that every manifold is an example of an orbifold, but not every orbifold is a manifold. In the example of the three-torus above, the different copies of yourself differ by translations, meaning that you can move one copy of yourself to another simply by shifting without reflecting or rotating either of the copies. The idea of an orbifold allows for visual copies of an inhabitant to be related not only by translations, but also by rotations and reflections, both of which leave some points of the space fixed. To imagine what the inside of an example of an orbifold "looks like", imagine the experience of being inside of a polyhedron where all of the faces are made of mirrors. Looking around, you will see many copies of yourself. Near an edge where two mirrors meet, you will see a number of copies of yourself obtained either by rotation about the edge or rotation about the edge followed by a reflection. In an orbifold, we think of all of these copies as actually being the same person, not a mirror image as a person having this experience would naturally assume. See Figure 2 to see the earth inside of a mirrored hyperbolic dodecahedron. Look closely to see that the two nearest "copies" of earth are mirror images of one another.

Based on this description alone, describing a three-dimensional orbifold seems complicated. Fortunately, there is a theorem that a three-dimensional orbifold can be completely described by specifying an underlying manifold, called the base space, and a graph sitting inside the base space, called the singular locus. In this context, a graph is a collection of points joined together by one-dimensional edges. The singular locus should be thought of as the points in the orbifold that correspond to where the mirrors meet in the example in the previous paragraph. At points away from the singular locus, an orbifold looks like a manifold. Near the singular locus, an orbifold looks like a hall of mirrors as described above. For a more in-depth, yet still largely nontechnical discussion of these ideas, see section 2 of the paper [9] by Thurston (Available temporarily at `http://personal.morris.umn.edu/~catkinso/thurston-how-to-see.pdf`).

1.3. **Shape and size in hyperbolic space:** Having now described the relevant background, I will now describe the main themes of my research. Currently, my main research project is working on finding techniques for determining the volume of hyperbolic 3–orbifolds in terms of their shape. Below I describe the idea behind this course of study.

For three-dimensional orbifolds that have hyperbolic geometry, shape determines size. Slightly more precisely, I mean that if the base space and singular locus of a hyperbolic orbifold are specified then the volume of the orbifold is determined by this data. We've replaced "shape" with "base space" and "singular locus" and replaced "size" with "volume." This means that the orbifold cannot be rescaled as discussed for hyperbolic triangles above. This is a consequence of what is known as the Mostow-Prasad Rigidity Theorem [8]. This connection goes both directions: If one knows that two hyperbolic orbifolds have different volume, then one also knows that they also have different shape. In practice, volume has proved to be the most effective quantity for distinguishing hyperbolic orbifolds.

The rigidity of hyperbolic space manifests itself in another way with respect to volume. A theorem of Dunbar and Meyerhoff says that the list of volumes of all hyperbolic orbifolds is a relatively nice list of numbers [6]. One consequence is that there are only finitely many orbifolds of a given volume and that the list of volumes occur in an organized fashion. Their theorem implies that there is a smallest possible volume for a hyperbolic orbifold. In 2012 Marshall and Martin identified the smallest volume hyperbolic three-dimensional orbifold [7]. Another consequence of the theorem of Dunbar and Meyerhoff is that in any family of hyperbolic three-dimensional orbifolds, there is a volume minimizer.

Although the volume is determined by shape, there is no practical technique to compute the volume of a given hyperbolic orbifold. In theory, there is a function that assigns the volume to a specified base space and singular locus, but in practice, determining the specifics of precisely how to determine the value of this function in terms of the base space and singular locus alone seems to be an intractable problem. There is a large body of research that looks into how to determine or at least estimate the volume in terms of the shape of an orbifold. In previous work, I have shown how to estimate the volume in the case where the orbifold arises from a polyhedron [1, 2]. In joint work with David Futer of Temple University, we have shown how to bound below the volume of various orbifolds that have no vertices in their singular loci [3]. More recently, Futer and I extended the work done in this paper to prove a theorem describing an infinite family of smallest volume examples [4]. Figure 3 is a view of one of these examples containing a single galaxy for reference. I reiterate that each "copy" of the galaxy is a different view of a single galaxy. Also, in joint work with Mallepalle, Melby, Rafalski, and Vaccaro (all but Rafalski were undergraduates at the time), we gave a technique for bounding the volume of an orbifold in terms of the topology of its guts [5].

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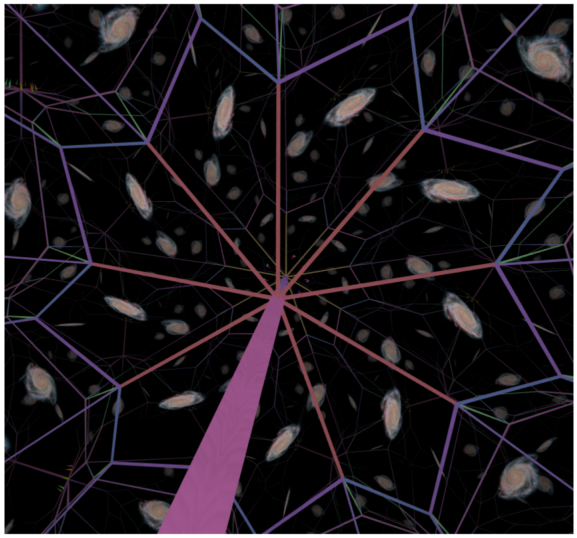

FIGURE 3. This is an interior view of the smallest volume hyperbolic orbifold having all torsion orders at least 9. We proved this in [4]. The colored edges are not part of the orbifold but are included in this figure to indicate the location of the singular points. Notice that there are 9 copies of the galaxy around the central pinkish axis.

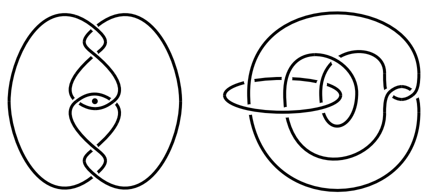

FIGURE 4. The knot 5_2 and the link 7_1^2.

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