Hyperbolic 3–orbifolds with high torsion

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background and motivation

3–orbifolds

An orientable 3–dimensional orbifold $\mathcal{O}$ is a space locally modeled on B^3/G where B^3 is a 3–ball and $G \subset \mathrm{SO}(3)$ is a finite group acting on B^3 by rotations.

An orientable 3–orbifold $\mathcal{O}$ can be described by specifying:

- an orientable 3–manifold $X_{\mathcal{O}}$ (the base space), and
- an embedded trivalent graph $\Sigma_{\mathcal{O}}$ with edges labeled by integers ≥ 2 (the singular locus).

3–orbifolds

$\Sigma_{\mathcal{O}}$ and its labeling specify the local groups G. The integer labeling an edge is called the torsion order of the edge.

A neighborhood of a point in an edge labeled n is the quotient of B^3 by $\mathbb{Z}_n$ acting by rotations.

A neighborhood of a vertex of $\Sigma_{\mathcal{O}}$ is the quotient of B^3 by an orientation preserving spherical triangle group.

Hyperbolic 3–orbifolds

A 3–orbifold is called hyperbolic if $\mathcal{O} = \mathbb{H}^3/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. Torsion in Γ gives rise to the singular locus.

What does this look like?

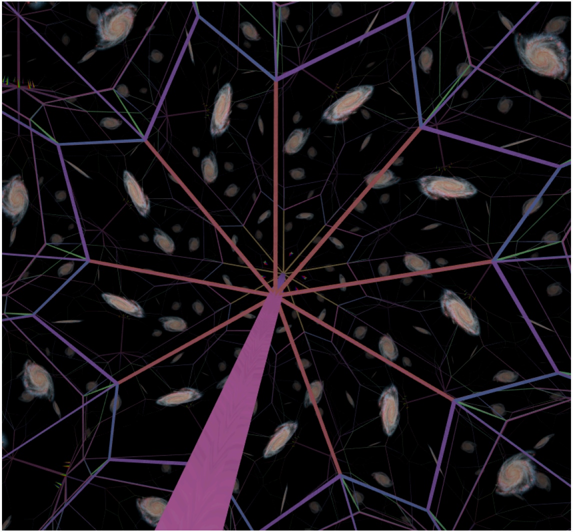

Figure 1: Looking along an edge labeled 9 in the hyperbolic structure on `m004`$(9, 0)$.

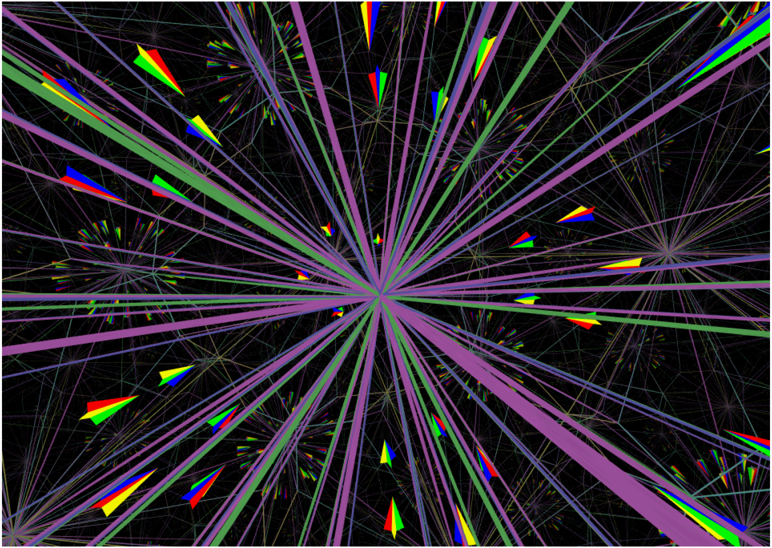

Figure 2: Looking towards a $(2, 3, 5)$ vertex in a doubled hyperbolic tetrahedron.

Hyperbolic 3–orbifolds

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Dunbar and Meyerhoff proved that volumes of hyperbolic 3–orbifolds form a closed, non–discrete, well–ordered subset of $\mathbb{R}^+$.

Motivation

Question

What is the hyperbolic 3*–orbifold of smallest volume?*

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Answer

Gehring-Marshall-Martin: The unique hyperbolic 3–orbifold of smallest volume, $\mathcal{O}_{min}$, is shown below. It's volume is 0.03905.

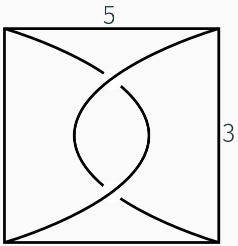


Figure 3: Base space is S^3. Unlabeled edges have 2–torsion.

Motivation

We'll give an answer a family of related questions:

Question

For an integer $n \geq 2$, what is the lowest volume hyperbolic 3–orbifold with all torsion orders bounded below by n?

Link orbifolds

Definition

For $n \geq 2$, let L_n be the set of all orientable hyperbolic 3–orbifolds with non–empty singular locus and with all torsion orders bounded below by n.

Lemma

For $n \geq 4$, the singular locus of every element of L_n is a closed 1–manifold.

We call these link orbifolds.

main results

Theorem (A-Futer, 2016)

Every element of L_3 has volume at least 0.2371.

Conjecture

The unique lowest–volume element of L_3 is the orbifold with base space S^3 and singular locus the knot 5_2 labeled 3.

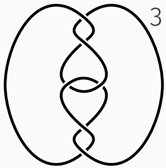
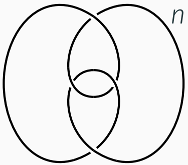


Figure 4: Conjectured volume minimizer in L_3 of volume $0.3142\cdots = \mathrm{Vol}(M_W)/3$. M_W is the Weeks manifold.

Main theorem

Theorem (A-Futer, 2016)
For all $n \geq 4$, the unique lowest–volume element of L_n is the orbifold $\mathcal{P}_n$, with base space S^3 and singular locus the figure–8 knot labeled n.



We'll focus on this theorem today.

Main theorem

Theorem (A-Futer, 2016)

For all $n \geq 4$, the unique lowest–volume element of L_n is the orbifold $\mathcal{P}_n$, with base space S^3 and singular locus the figure–8 knot labeled n.

n	$\mathrm{Vol}(\mathcal{P}_n)$
4	0.50747080320
5	0.93720685476
6	1.22128745890
7	1.41175465336
8	1.54386327614
9	1.6386006808
∞	2.0298832128

Table 1: Vesnin and Mednykh have formula for $\mathrm{Vol}(\mathcal{P}_n)$ in terms of Lobachevsky function.

An application to manifolds with symmetry

Theorem (A-Futer, 2016)

Let M be an orientable hyperbolic 3–manifold of finite volume and let G be a group of orientation preserving isometries of M. Let p be the smallest prime dividing $|G|$, or else 1 if G is trivial. Then

$$\mathrm{Vol}(M) \geq |G| \cdot w_p, \text{ where } w_p = \begin{cases} \mathrm{Vol}(\mathcal{O}_{min}) \geq 0.03905 & p = 2 \\ 0.2371 & p = 3 \\ \mathrm{Vol}(\mathcal{P}_5) \geq 0.9372 & p = 5 \\ \mathrm{Vol}(M_W) \geq 0.9427 & \text{otherwise.} \end{cases}$$

elements of proof

Main theorem

Theorem (A-Futer, 2016)
For all $n \geq 4$, the unique lowest–volume element of L_n is the orbifold $\mathcal{P}_n$, with base space S^3 and singular locus the figure–8 knot labeled n.

Proof outline

1. Show that $M_{\mathcal{O}} = \mathcal{O} \setminus \Sigma_{\mathcal{O}}$ is one of a finite list of cusped hyperbolic 3–manifolds.
2. For $n \leq 14$, minimizer must be one of finitely many orbifold Dehn fillings of the finitely many manifolds identified in step (1). Rigorous computer search for volume minimizer.
3. For $n \geq 15$, use analytic estimates on the change in volume when filling one of the candidate cusped manifolds identified in (1).

1. Topological reduction

1. Show that $M_{\mathcal{O}} = \mathcal{O} \setminus \Sigma_{\mathcal{O}}$ is one of a finite list of cusped hyperbolic 3–manifolds.
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Let $M_{\mathcal{O}} = \mathcal{O} \setminus \Sigma_{\mathcal{O}}$.

Theorem

Suppose that $n \geq 4$ and that $\mathcal{O} \in L_n$ is a volume minimizer. If $n \neq 4, 8$, then $M_{\mathcal{O}}$ is one of `m003` *or* `m004`.

If $n = 4, 8$, then $M_{\mathcal{O}} \in \mathcal{C}$ where

$$\mathcal{C} = \{\mathtt{m003}, \mathtt{m004}, \mathtt{m006}, \mathtt{m007}, \mathtt{m009}, \mathtt{m010}, \mathtt{m011}, \mathtt{m015}, \mathtt{m016}, \mathtt{m017}\}.$$

(`mXXX` *are SnapPea census manifolds)*

1. Topological reduction ($M_{\mathcal{O}} \in \mathcal{C}$)

Theorem is a consequence of a theorem of Gabai, Meyerhoff, and Milley.

Theorem

If M is a one–cusped hyperbolic 3–manifold with $\mathrm{Vol}(M) \leq 2.848$, then $M \in \mathcal{C}$.

We show that if $\mathcal{O}$ minimizes volume in L_n, then $\mathrm{Vol}(M_{\mathcal{O}}) \leq 2.848$.

1. Topological reduction ($\mathrm{Vol}(M_{\mathcal{O}}) \leq 2.848$)

Combining work of Gehring, Marshall, and Martin giving lower bounds on the collar radius around singular locus in terms of torsion order with work of Agol and Dunfield controlling the change in volume when drilling, we obtain

Proposition

$$\mathrm{Vol}(\mathcal{O}) \geq \frac{(x_n^2 - 1)^{3/2}}{x_n^3} \left(1 + \frac{0.91}{x_n}\right)^{-1} \cdot \mathrm{Vol}(M_{\mathcal{O}}),$$

where $x_n = \cosh(2r_n)$ *and* r_n *is the collar radius.*

1. Topological reduction ($\mathrm{Vol}(M_{\mathcal{O}}) \leq 2.848$)

Combining work of Gehring, Marshall, and Martin giving lower bounds on the collar radius around singular locus in terms of torsion order with work of Agol and Dunfield controlling the change in volume when drilling, we obtain

Proposition

$$\mathrm{Vol}(\mathcal{O}) \;\geq\; f(n) \cdot \mathrm{Vol}(M_{\mathcal{O}}),$$

where $x_n = \cosh(2r_n)$ *and* r_n *is the collar radius.*

1. Topological reduction ($\mathrm{Vol}(M_{\mathcal{O}}) \leq 2.848$)

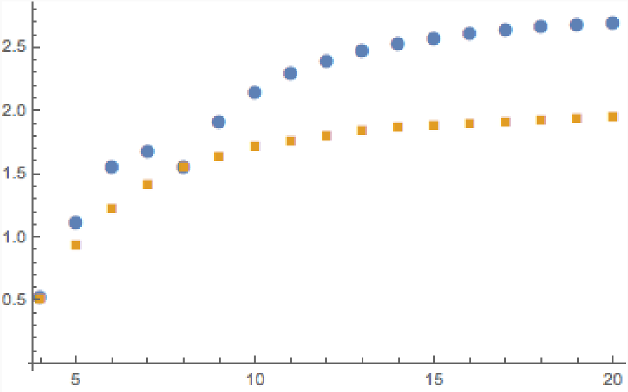

Figure 5: ● $= 2.848 \cdot f(n)$, ■ $= \mathrm{Vol}(\mathcal{P}_n)$. We know that $\mathrm{Vol}(\mathcal{O}) > 2.848 \cdot f(n)$ if $\mathrm{Vol}(M_{\mathcal{O}}) > 2.848$.

If the volume of $M_{\mathcal{O}} > 2.848$, then the volume of $\mathcal{O}$ is greater than that of $\mathcal{P}_n$.

Proof outline

1. Show that $M_{\mathcal{O}} = \mathcal{O} \setminus \Sigma_{\mathcal{O}}$ is one of a finite list of cusped hyperbolic 3–manifolds.
2. For $n \leq 14$, volume minimizer must be one of finitely many orbifold Dehn fillings of the manifolds identified in step (1). Rigorous computer search for volume minimizer.
3. For $n \geq 15$, use analytic estimates on the change in volume when filling one of the candidate cusped manifolds identified in (1).

2. $n \leq 14$. Computer search through finitely many orbifolds

If $\mathrm{Vol}(\mathcal{O}) \leq \mathrm{Vol}(\mathcal{P}_n)$, then $M_{\mathcal{O}} \in \mathcal{C}$ and $\mathcal{O}$ is a Dehn filling of $M_{\mathcal{O}}$.

For each $M = M_{\mathcal{O}}$, a theorem of Futer, Kalfagianni, and Purcell implies that the Dehn filling coefficients (a, b) of the filling slope $s = a\lambda + b\mu$ must lie in the set

$$\mathcal{F}_M^n := \left\{ (a, b) \in \mathbb{Z}^2 \; : \; 1 - \left(\frac{2\pi}{\ell(s)} \right)^2 \leq \left(\frac{\mathrm{Vol}(\mathcal{P}_n)}{\mathrm{Vol}(M)} \right)^{2/3} \right\}$$

This is a finite set.

2. $n \leq 14$. Computer search through finitely many orbifolds

For each of the 205 manifold–slope pairs, we use Snap and code developed by Moser and Milley to rigorously show that each filling has volume greater than that of $\mathcal{P}_n$ (except for $\mathcal{P}_n$ itself).

Proof outline

1. Show that $M_{\mathcal{O}} = \mathcal{O} \setminus \Sigma_{\mathcal{O}}$ is one of a finite list of cusped hyperbolic 3–manifolds.
2. For $n \leq 14$, minimizer must be one of finitely many orbifold Dehn fillings of the manifolds identified in step (1). Rigorous computer search for volume minimizer.
3. For $n \geq 15$, use analytic estimates on the change in volume when filling one of the candidate cusped manifolds identified in (1).

3. $n \geq 15$. Analytic estimates on change in volume when filling.

For each $n \geq 15$, the minimal volume element of L_n is a Dehn filling of either `m003` or `m004`. Note `m003` and `m004` both have the same volume of $2.02988\ldots$.

For a Dehn filling slope s on a cusped manifold M yielding a hyperbolic orbifold, define

$$\Delta V(s) = \mathrm{Vol}(M) - \mathrm{Vol}(M(s)).$$

The proof of the main theorem is completed by proving the following:

Proposition

Fix $n \geq 15$. Among all Dehn fillings of `m003` *or* `m004` *that produce link orbifolds with n–torsion, the largest value of $\Delta V(s)$ is realized by* $\mathtt{m004}(n, 0) = \mathcal{P}_n$.

To prove the proposition, we use the following theorem of Hodgson and Kerckhoff.

Theorem

Let M a cusped hyperbolic 3–manifold. Let s be a slope on some cusp of M, satisfying $A(s) \leq f(1/\sqrt{3}) = 0.68653\ldots$. Then $M(s)$ is hyperbolic and

$$\int_{\tilde{z}}^{1} l(z)\,dz \leq \Delta V(s) \leq \int_{\hat{z}}^{1} u(z)\,dz\,,$$

where $\hat{z} = f^{-1}(A(s))$ and $\tilde{z} = \tilde{f}^{-1}(A(s))$.

If you're wondering about a few definitions here...

3. $n \geq 15$. Oh no!

...I think this will make you stop wondering.

$$A(s) = \frac{(2\pi)^2 \mathrm{Area}(\partial C)}{\ell(s)^2},$$

$$l(z) = \frac{3.3957\, z^2 (z^2 - 3)(z^4 + 4z^2 - 1)}{2\,(z^4 - 6z^2 + 1)(z^2 + 1)^2},$$

$$u(z) = \frac{3.3957\, z^2 (z^4 + 4z^2 - 1)}{2\,(z^2 + 1)^3},$$

$$f(z) = \frac{3.3957(1 - z)}{\exp\left(\int_1^z F(w)\, dw\right)} \quad \text{where} \quad F(z) = -\frac{z^4 + 6z^2 + 4z + 1}{(z + 1)(z^2 + 1)^2},$$

$$\tilde{f}(z) = \frac{3.3957(1 - z)}{\exp\left(\int_1^z \tilde{F}(w)\, dw\right)} \quad \text{where}$$

$$\tilde{F}(z) = -\frac{z^6 + 7z^4 + 12z^3 - 9z^2 - 4z + 1}{(z + 1)(z^2 + 1)(z^2 + 2z - 1)(z^2 - 2z - 1)}.$$

3. $n \geq 15$. Analytic estimates on change in volume when filling.

Let $\tilde{s}$ be the filling of `m004` yielding $\mathcal{P}_n$.

We show that the lower bound from the Hodgson–Kerckhoff theorem on $\Delta V(\tilde{s})$ is larger than the upper bound on $\Delta V(s)$ for any competing filling.

This involves a careful analysis of the functions that shall not be shown again.

$$\int_{\tilde{z}}^{1} l(z)\, dz \leq \Delta V(s) \leq \int_{\hat{z}}^{1} u(z)\, dz\,,$$

Thank you.