Guts and volume for a simple family of hyperbolic 3–orbifolds

Chris Atkinson
University of Minnesota, Morris

The Topology of 3- and 4-Manifolds

Joint work with:
Jessica Mallepalle (Arcadia University)
Joseph Melby (University of Minnesota Morris)
Shawn Rafalski (Fairfield University)
Jennifer Vaccaro (Olin College of Engineering)

Joint work with:
Jessica Mallepalle (Arcadia University)
Joseph Melby (University of Minnesota Morris)
Shawn Rafalski (Fairfield University)
Jennifer Vaccaro (Olin College of Engineering)

Undergraduates. Worked out some preliminaries in an REU led by Shawn.

Theme

How does topological complexity affect the volume of a hyperbolic 3–orbifold?

Theme

How does topological complexity affect the volume of a hyperbolic 3–orbifold?

We'll see how to get a lower bound on the volume of a hyperbolic 3–orbifold containing an incompressible 2–suborbifold by understanding the topology of the complement of the 2–suborbifold.

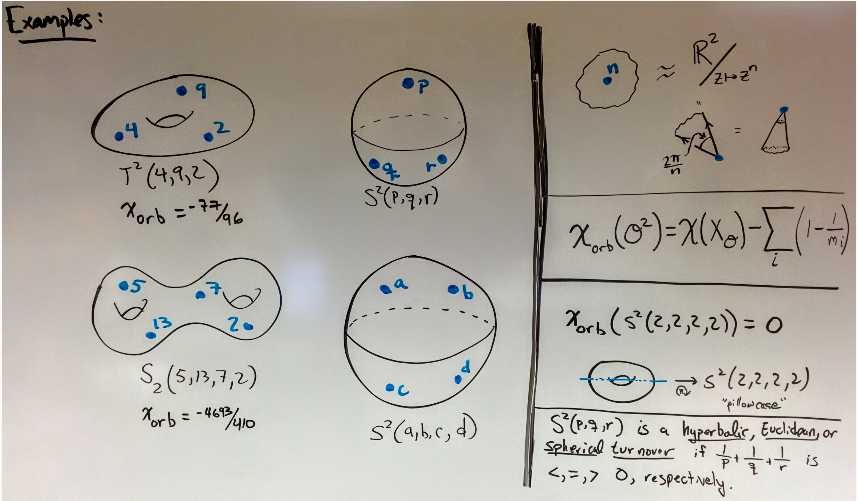
Background

2–orbifolds

A 2–dimensional orbifold $\mathcal{O}$ is a space locally modeled on B^2/G where B^2 is a 2–ball and G is a (possibly trivial) finite group acting on B^2 by rotations or reflections.

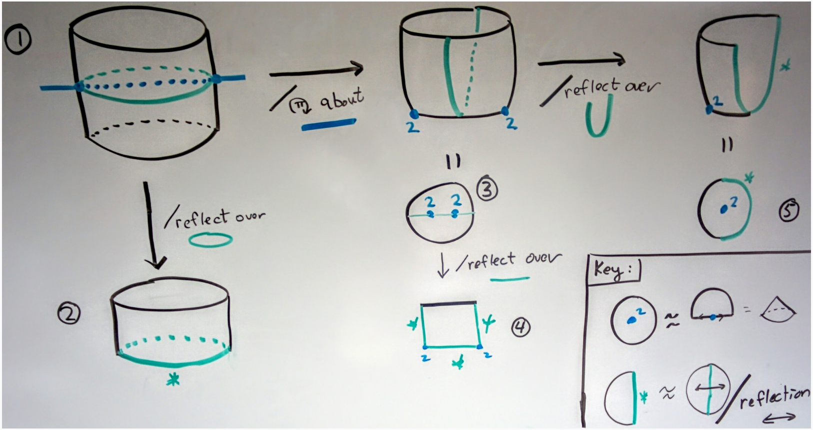
We'll mainly be concerned with orientable 2–orbifolds. In this case, G consists only of rotations.

2-orbifolds


Examples:
T²(4,9,2)
χorb = -77/96
S²(p,q,r)
S₂(5,13,7,2)
χorb = -4693/410
S²(a,b,c,d)
≈ R²/z↦zⁿ
2π/n
χorb(O²) = χ(X_O) − Σ_i (1 − 1/m_i)
χorb(S²(2,2,2,2)) = 0
S²(2,2,2,2)
"pillowcase"
S²(p,q,r) is a hyperbolic, Euclidean, or spherical turnover if 1/p + 1/q + 1/r is <, =, > 0, respectively.

Orbifold annuli

There are 5 kinds of orbifold annuli (quotients of a standard $S^1 \times I$).



3–orbifolds

An orientable 3–dimensional orbifold $\mathcal{O}$ is a space locally modeled on B^3/G where B^3 is a 3–ball and $G \subset \mathrm{SO}(3)$ is a finite group acting on B^3 by rotations.

An orientable 3–orbifold $\mathcal{O}$ can be described by specifying:

- an orientable 3–manifold $X_{\mathcal{O}}$ (the base space), and
- an embedded trivalent graph $\Sigma_{\mathcal{O}}$ with edges labeled by integers ≥ 2 (the singular locus).

3–orbifolds

$\Sigma_{\mathcal{O}}$ and its labeling specify the local groups G. The integer labeling an edge is called the torsion order of the edge.

A neighborhood of a point in an edge labeled n is the quotient of B^3 by $\mathbb{Z}_n$ acting by rotations.

A neighborhood of a vertex of $\Sigma_{\mathcal{O}}$ is the quotient of B^3 by an orientation preserving spherical triangle group.

Hyperbolic 3–orbifolds

A 3–orbifold is called hyperbolic if $\mathcal{O} = \mathbb{H}^3/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. Torsion in Γ gives rise to the singular locus.

What does this look like?

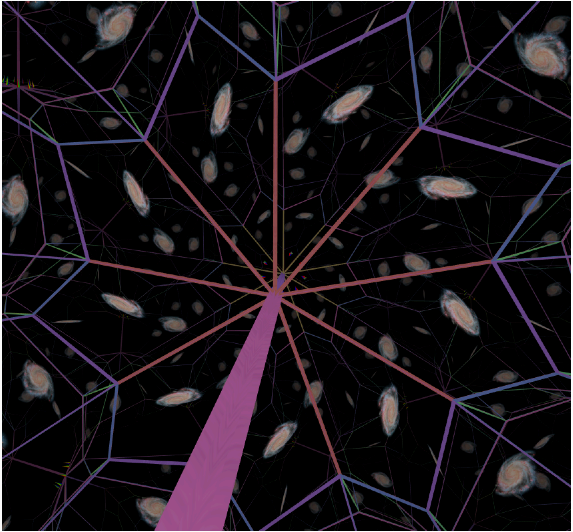


Figure 1: Looking along an edge labeled 9 in the hyperbolic structure on `m004`$(9, 0)$.

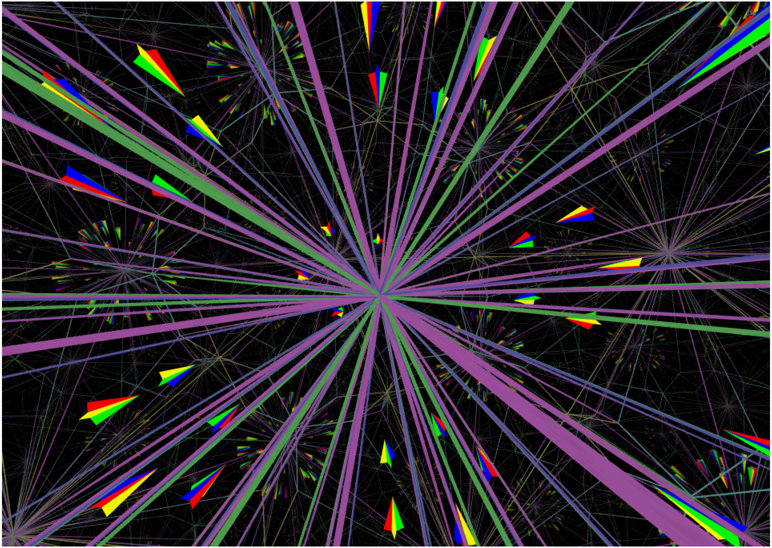


Figure 2: Looking towards a $(2, 3, 5)$ vertex in a doubled hyperbolic tetrahedron.

main result

The idea of the statement of the theorem

"Theorem"

Let $\mathcal{O}$ be compact, orientable, hyperbolic 3–orbifold with base space S^3. Let $\mathcal{S}$ be an incompressible 2–suborbifold of $\mathcal{O}$ with base space S^2 and cone points having torsion orders n_i for $i \in \{1, \ldots, k\}$.

Then the volume of $\mathcal{O}$ is bounded below by a function of the n_i and the topology of $\mathcal{O} \backslash\backslash \mathcal{S}$.

$\mathcal{O} \backslash\backslash \mathcal{S}$ is the path metric completion of $\mathcal{O} \setminus \mathcal{S}$.

Main theorem

"Theorem"

Let $\mathcal{O}$ be compact, orientable, hyperbolic 3–orbifold with base space S^3. Let $\mathcal{S}$ be an incompressible 2–suborbifold of $\mathcal{O}$ with base space S^2 and cone points having torsion orders n_i for $i \in \{1, \ldots, k\}$.

Then the volume of $\mathcal{O}$ is bounded below by a function of the n_i and the topology of $\mathcal{O} \backslash\backslash \mathcal{S}$.

In practice, this function can be computed by hand for any given example. We show how to detect the guts of $\mathcal{O} \backslash\backslash \mathcal{S}$ which in turn indicates how to compute the function.

Actual theorem

Theorem

In the case where there are four cone points, the volume bound is one of the following. The cases depend on the topology of $\mathcal{O} \backslash\backslash \mathcal{S}$.

1. $\mathrm{Vol}(\mathcal{O}) \geq -V_8 \chi_{orb}(\mathcal{S}) = V_8 \left(2 - \sum_{i=1}^{4} 1/n_i\right)$, *or*
2. $\mathrm{Vol}(\mathcal{O}) \geq \frac{1}{2} V_8 \left(-\chi_{orb}(\mathcal{S}) + 1 - 1/n_{i_1} - 1/n_{i_2}\right)$, *(where $\{n_{i_1}, n_{i_2}\} \subset \{n_1, n_2, n_3, n_4\}$), or*
3. $\mathrm{Vol}(\mathcal{O}) \geq -\frac{1}{2} V_8 \chi_{orb}(\mathcal{S})$, *or*
4. $\mathrm{Vol}(\mathcal{O}) \geq \frac{1}{2} V_8 \left(2 - 1/n_{i_1} - 1/n_{i_2} - 1/n_{i_3} - 1/n_{i_4}\right)$, *(where $\{n_{i_1}, n_{i_2}, n_{i_3}, n_{i_4}\} \subset \{n_1, n_2, n_3, n_4\}$), or*
5. $\mathrm{Vol}(\mathcal{O}) \geq \frac{1}{2} V_8 \left(1 - 1/n_{i_1} - 1/n_{i_2}\right)$, *(where $\{n_{i_1}, n_{i_2}\} \subset \{n_1, n_2, n_3, n_4\}$), or*
6. *$\mathcal{O} \backslash\backslash \mathcal{S}$ is hungry.*

Theorem (Agol-Storm-Thuston 2005)
Let $\mathcal{O}$ be a closed, hyperbolic 3–orbifold and $S \subset \mathcal{O}$ be an orbifold incompressible 2–suborbifold. Then

$$\mathrm{Vol}(\mathcal{O}) \geq -V_8 \chi_{orb}(\mathrm{Guts}(\mathcal{O} \backslash\backslash S)).$$

$V_8 \approx 3.66386$ is the volume of the regular, ideal, hyperbolic octahedron. $\mathcal{O} \backslash\backslash S$ is the path metric completion of the complement of S in $\mathcal{O}$.

Guts of $\mathcal{O} \backslash\backslash \mathcal{S}$ (hand wavy)

$\mathcal{O}$ is a hyperbolic 3–orbifold containing an incompressible 2–suborbifold $\mathcal{S}$.

$\mathcal{O} \setminus \mathcal{S}$ has an incomplete hyperbolic structure obtained by just deleting $\mathcal{S}$.

Can we complete $\mathcal{O} \setminus \mathcal{S}$ to a hyperbolic orbifold with totally geodesic boundary?

Guts of $\mathcal{O} \backslash\backslash \mathcal{S}$ (hand wavy)

$\mathcal{O}$ is a hyperbolic 3–orbifold containing an incompressible 2–suborbifold $\mathcal{S}$.

$\mathcal{O} \setminus \mathcal{S}$ has an incomplete hyperbolic structure obtained by just deleting $\mathcal{S}$.

Can we complete $\mathcal{O} \setminus \mathcal{S}$ to a hyperbolic orbifold with totally geodesic boundary?

Not always. If we could, then its double along $\mathcal{S}$ would be atoroidal, so we need $\mathcal{O} \backslash\backslash \mathcal{S}$ to be acylindrical.

$\mathrm{Guts}(\mathrm{O} \backslash\backslash \mathcal{S})$ is obtained by throwing away the problematic parts that obstruct this completion.

Guts

$\mathcal{O}$ is a hyperbolic 3–orbifold.

$N = \mathrm{Guts}(\mathcal{O})$ is a codimension 0 suborbifold such that $\partial N = \partial_0 N \cup \partial_1 N$ where

- $\partial_0 N = N \cap \partial\mathcal{O}$,
- $\partial_1 N$ consists of annuli and tori with $\partial\partial_1 N = \partial_1 N \cap \partial\mathcal{O}$, and
- $(N, \partial_1 N)$ is the maximal pared acylindrical suborbifold such that no components of N are solid orbifold tori.

Guts

$\mathcal{O}$ is a hyperbolic 3–orbifold.

$N = \mathrm{Guts}(\mathcal{O})$ is a codimension 0 suborbifold such that $\partial N = \partial_0 N \cup \partial_1 N$ where

- $\partial_0 N = N \cap \partial\mathcal{O}$,
- $\partial_1 N$ consists of annuli and tori with $\partial\partial_1 N = \partial_1 N \cap \partial\mathcal{O}$, and
- $(N, \partial_1 N)$ is the maximal pared acylindrical suborbifold such that no components of N are solid orbifold tori.

Can replace third condition with: $\mathrm{Guts}(\mathcal{O}) \setminus \partial_1 \mathrm{Guts}(\mathcal{O})$ admits a complete hyperbolic structure with totally geodesic boundary and the double of $\mathcal{O} \setminus N$ is a graph orbifold.

Guts

$\mathcal{O}$ is a hyperbolic 3–orbifold.

$N = \mathrm{Guts}(\mathcal{O})$ is a codimension 0 suborbifold such that $\partial N = \partial_0 N \cup \partial_1 N$ where

- $\partial_0 N = N \cap \partial\mathcal{O}$,
- $\partial_1 N$ consists of annuli and tori with $\partial\partial_1 N = \partial_1 N \cap \partial\mathcal{O}$, and
- $(N, \partial_1 N)$ is the maximal pared acylindrical suborbifold such that no components of N are solid orbifold tori.

Can replace third condition with: $\mathrm{Guts}(\mathcal{O}) \setminus \partial_1 \mathrm{Guts}(\mathcal{O})$ admits a complete hyperbolic structure with totally geodesic boundary and the double of $\mathcal{O} \setminus N$ is a graph orbifold.

For a cheap laugh, I'm saying that $\mathcal{O} \setminus\setminus \mathcal{S}$ is hungry if it has empty guts. (In the context of manifolds, if the guts of $M \setminus\setminus \mathcal{S}$ is empty, then $\mathcal{S}$ is called a fibroid.)

Pared orbifolds

A pared orbifold is a pair $(\mathcal{O}, P)$, where $\mathcal{O}$ is a compact, orientable, irreducible 3-orbifold and $P \subset \partial\mathcal{O}$ is a union of essential orbifold annuli and tori (possibly empty) such that

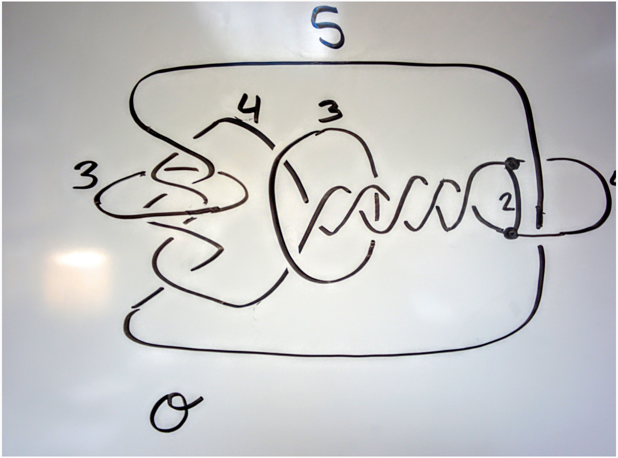
1. every abelian, noncyclic subgroup of $\pi_1(\mathcal{O})$ is conjugate to a subgroup of the fundamental group of a component of P, and
2. every map of an orbifold annulus $(A, \partial A) \to (\mathcal{O}, P)$ that is π_1-injective deforms, as a map of pairs, into P.

P is called the parabolic locus of $(\mathcal{O}, P)$, and we define $\partial_0\mathcal{O}$ to be $\partial\mathcal{O} - \mathrm{int}(\mathrm{P})$.

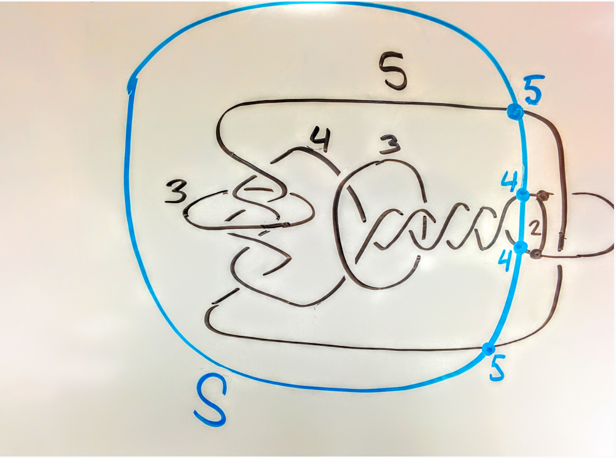
To prove the main theorem, we look at $\mathcal{O} \setminus\setminus \mathcal{S}$ and show how to find the annuli that appear in $\partial_1 \mathrm{Guts}(\mathcal{O} \setminus\setminus \mathcal{S})$. The volume bounds then come from applying Agol-Storm-Thurston and thinking about how the various boundary components of the guts can contribute to the volume.

We also give a complete characterization of what the orbifold $\mathcal{O} \setminus\setminus \mathcal{S}$ looks like in the case of empty guts.

An example

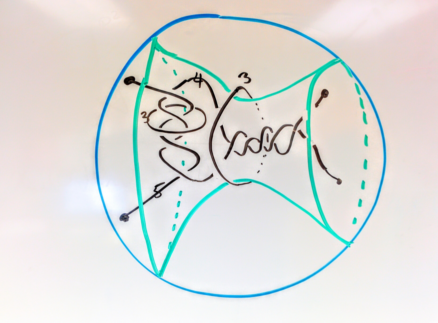


An example



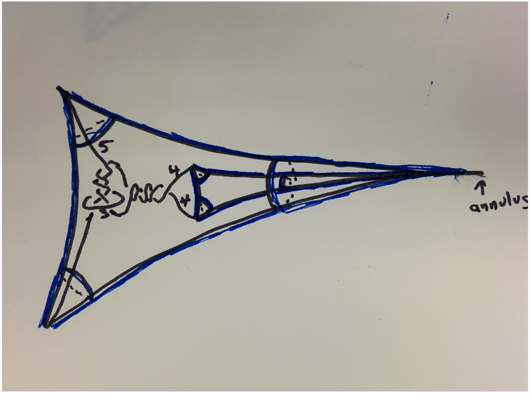
$\mathcal{S}$ is an incompressible 2–suborbifold.

An example



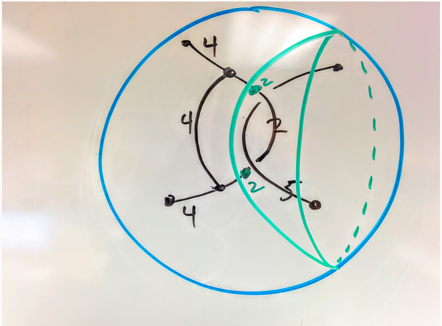
One component of $\mathcal{O} \backslash\backslash \mathcal{S}$. This is essentially the only way that a non-singular annulus (in green) can arise.

An example



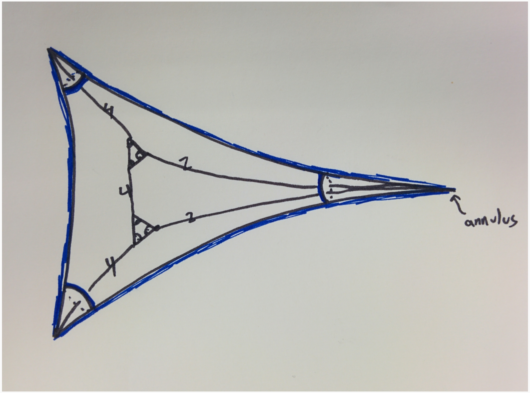
This is the guts of the picture on the previous slide. It's Euler characteristic is $-\frac{11}{10}$, so this portion contributes $-\frac{1}{2}\left(\frac{-11}{10}\right)V_8 \approx 2.015$ to the volume bound.

An example



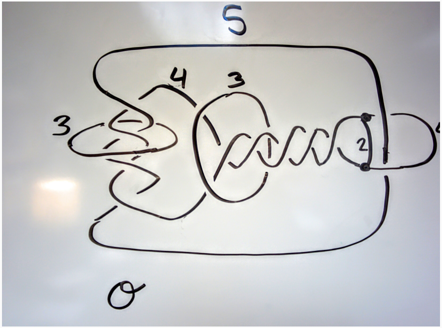
The other component of $\mathcal{O} \backslash\backslash \mathcal{S}$. The green surface is a singular annulus. The portion on the right is an orbifold I–bundle, so is not part of the guts.

An example



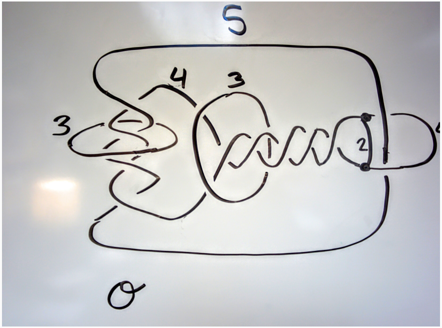
This is the guts of the picture on the previous slide. It has Euler characteristic $-\frac{1}{2}$, so contributes $-\frac{1}{2}\left(\frac{-1}{2}\right)V_8 \approx 0.915$ to the volume bound.

An example



Summing the bounds from either side of $\mathcal{S}$, we find that $\mathrm{Vol}(\mathcal{O}) \geq 2.015$.

An example



Summing the bounds from either side of $\mathcal{S}$, we find that $\mathrm{Vol}(\mathcal{O}) \geq 2.015$. According to Orb, the actual volume of $\mathcal{O}$ is about 12.3.

Thank you.