Small volume link orbifolds

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Joint work with David Futer (Temple University)

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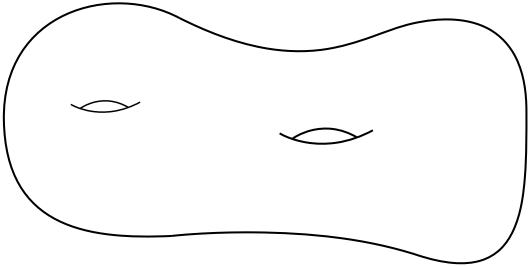
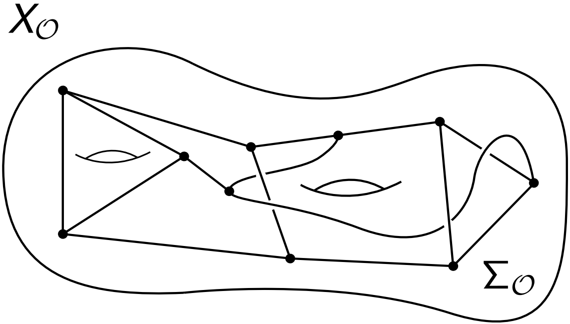
What is a 3-orbifold?



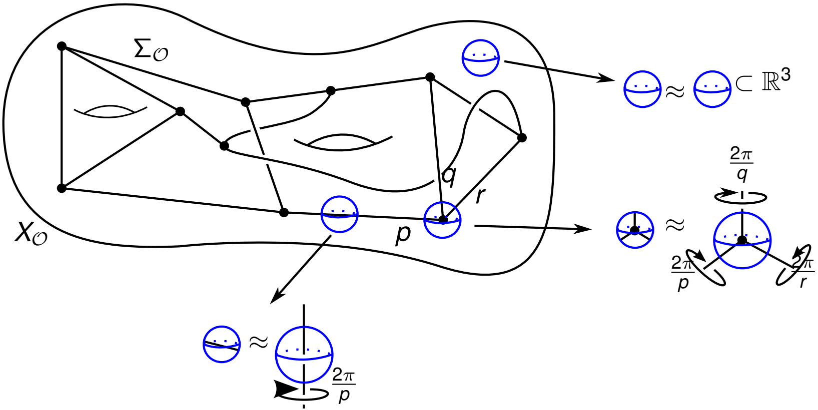
Base space = $X_{\mathcal{O}}$ = Closed, orientable, three-dimensional manifold.

What is a 3-orbifold?



Singular locus = $\Sigma_{\mathcal{O}}$ = trivalent graph labeled by natural numbers ≥ 2 embedded in $X_{\mathcal{O}}$.

What is 3-orbifold?



$(p, q, r) \in \{(2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)\}$

- $\mathcal{O} = (X_{\mathcal{O}}, \Sigma_{\mathcal{O}})$

Mostow–Prasad and Dunbar–Meyerhoff

- A 3–orbifold $\mathcal{O}$ is hyperbolic if $\mathcal{O} = \mathbb{H}^3/\Gamma$ for some discrete subgroup $\Gamma \leq \text{Isom}(\mathbb{H}^3)$.

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$$(X_{\mathcal{O}}, \Sigma_{\mathcal{O}}) \mapsto \mathrm{Vol}_{\mathbb{H}^3}(\mathcal{O})$$

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- Dunbar–Meyerhoff implies that in any subclass of finite–volume hyperbolic 3–orbifolds, we can find a member of smallest volume.
- Motivating question:
  What is the smallest <insert adjective here> hyperbolic 3–orbifold?

What is the smallest volume hyperbolic 3–orbifold?

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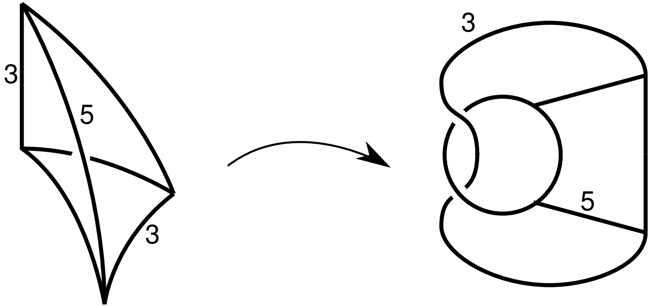
Marshall–Martin (2012) identified the smallest volume hyperbolic 3–orbifold. It has volume 0.03905 . . . .
Builds on a large body of work of subsets of {Gehring, Machlachlan, Marshall, Martin, Reid}



What is the smallest volume hyperbolic 3–orbifold having no singular locus?

What is the smallest volume hyperbolic 3–manifold?

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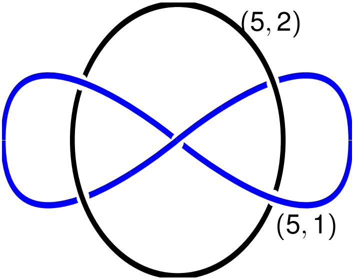
Gabai–Meyerhoff–Milley have proved that the Weeks manifold is the smallest volume hyperbolic 3–manifold. It has volume 0.9427...



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- A link orbifold is one with no vertices in its singular locus.
- Gehring, Marshall, and Martin showed that a link orbifold must have volume at least 0.041.
- They speculated that the smallest volume link orbifold should have significantly larger volume.

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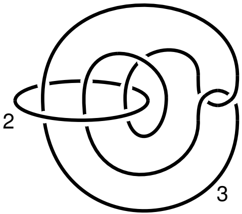
Conjecture (A-Futer)

The link orbifold pictured is the unique hyperbolic link orbifold of minimal volume. Its volume is 0.1571 . . . .

  - This volume is one–sixth the volume of the Weeks manifold.

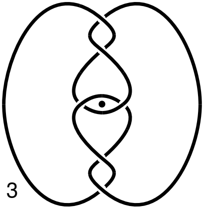


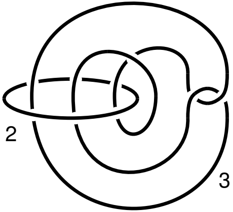
Results

Theorem (A-Futer (2013))

If $\mathcal{O}$ has

- *singular locus a knot with base space the* 3–*sphere, then* $\mathrm{Vol}(\mathcal{O}) \geq 0.31423...$
- *singular locus a link with base space the* 3–*sphere, then* $\mathrm{Vol}(\mathcal{O}) \geq 0.15711...$





Results

Theorem (A-Futer (2013))

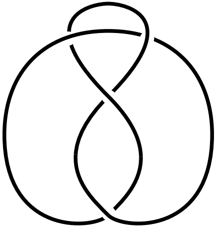
If $\mathcal{O}$ is a link orbifold with 2–torsion null–homologous, then $\mathrm{Vol}(\mathcal{O}) \geq 0.1185$.

Higher torsion

Let $\mathcal{L}_n$ be the set of hyperbolic link orbifolds with all torsion orders at least n.

Theorem (A-Futer (2014 in preparation))

Let n be 4, 6, or ≥ 8 and let $\mathcal{O} \in \mathcal{L}_n$. Then the volume of $\mathcal{O}$ is at least the volume of the figure figure–8 knot in S^3, labeled n.



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n	Volume
4	0.50747080320
6	1.22128745890
8	1.54386327614
9	1.6386006808
10	1.7085709483
11	1.76158141128
12	1.8026332233
13	1.83503265952
14	1.86102867909

Main techniques:

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Drill

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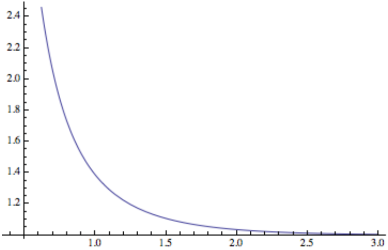
- For $n = 4, 6, 8$ and if $\mathrm{Vol}(M_{\mathcal{O}})$ is big (≥ 2.848), use Agol–Dunfield:

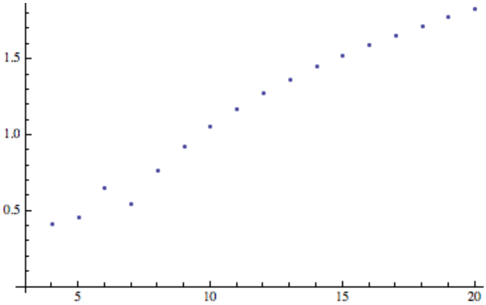
$$\frac{\mathrm{Vol}(M_{\mathcal{O}})}{\mathrm{Vol}(\mathcal{O})} \leq f(R)$$

where R is the (maximal) radius of an embedded collar of the singular locus.



Drill

- Gehring–Martin: lower bounds on collar radius in terms of n
$$R \geq c(n)$$

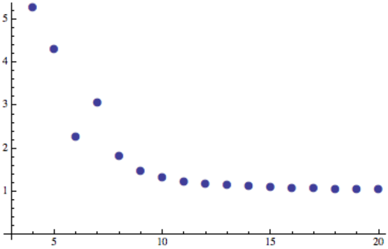


Drill

- Combine Agol–Dunfield and Gehring–Martin:

Theorem (Agol-Dunfield + Gehring-Martin)

$$\frac{\mathrm{Vol}(M_{\mathcal{O}})}{\mathrm{Vol}(\mathcal{O})} \leq f(n)$$



Drill

- If $n \geq 9$ and $\mathrm{Vol}(M_{\mathcal{O}})$ is big, we use a remark of Hodgson–Masai (an application of Hodgson–Kerckhoff):

$$-7.05 \leq \frac{\pi^2}{\mathrm{Vol}(M_{\mathcal{O}}) - \mathrm{Vol}\mathcal{O}} - Q(a,b) \leq 5.82$$

($Q(a,b)$ is the normalized length of the filling slope...)

Fill

- If $\mathrm{Vol}(M_{\mathcal{O}}) < 2.848$, then Gabai–Meyerhoff–Milley's mom technology tells us that $M_{\mathcal{O}}$ is one of ten SnapPea census manifolds.

$$M_{\mathcal{O}} \in \{\texttt{m003}, \texttt{m004}, \texttt{m006}, \texttt{m007}, \texttt{m009}, \texttt{m010}, \texttt{m011}, \texttt{m015}, \texttt{m016}, \texttt{ or m017}\}$$

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- In order for $\mathrm{Vol}(\mathcal{O})$ to have volume less than our target volume, Futer–Kalfagianni-Purcell tells us that the filling slope has to be one of finitely many possibilites.

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- Use Snap to construct triangulations on each filling.

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- Use Snap to construct triangulations on each filling.
- Use Milley's implementation of Moser's algorithm to rigorously check that all (but one) of these fillings is actually hyperbolic and has volume above the target volume.

(also)

many special cases that have no place in a 15 minute talk...

Thank you!

감사합니다